Note: minimize problem is equivalent to the maximize one.

Example: $\min\ z = 40x_1 + 36x_2$

$$\begin{aligned} x_1 &\leq 8 \\ \text{s.t}\quad x_2 &\leq 10 \\ 5x_1 + 3x_2 &\geq 45 \\ x_1, x_2 &\geq 0 \end{aligned}$$

It is equivalent to the following problem

$$\max\ z' = -40x_1 - 36x_2$$

$$\begin{aligned} x_1 &\leq 8 \\ \text{s.t}\quad x_2 &\leq 10 \\ 5x_1 + 3x_2 &\geq 45 \\ x_1, x_2 &\geq 0 \end{aligned}$$

The optimal solution to both problems will be the same while their optimal values will differ by a minus sign. In other words, the Min $z = -$ (Max z').

Duality in Linear Programming

Associated with every linear programming problem, called the primal (or original), there is another linear programming problem called its dual. These two problems possess very interesting and closely related properties. If the optimal solution to any one is known, the optimal solution to the other can readily be obtained.

1- The objective function coefficient of the primal problem becomes the R.H.S constraint of the dual.
2- Similarly, the R.H.S. constant of the primal becomes the cost coefficient of the dual.
3- The inequalities are reversed in the constraints.
4- The objective is changed from Max in primal into Min in the dual.
5- Each column in the primal corresponding constraint (row) in the dual.
6- Each constraint (row) in the primal corresponds to a column in the dual. Hence there is one dual variable for every constraint.
7- The dual of the dual is primal.

Note: If the primal is maximization, then the dual is minimization, and vice versa.

A linear programming is said to be in symmetric form if all the variables are restricted to be non-negative, and all the constraints are inequalities (in a maximization problem the inequalities must be less than or equal to form while in a minimization problem must be greater than or equal to form.

Symmetric Dual LP:

We shall now give the general representation of the primal-dual problems in symmetric form.

Primal

$$\text{maximize } z = c_1x_1 + c_2x_2 + \cdots + c_nx_n$$

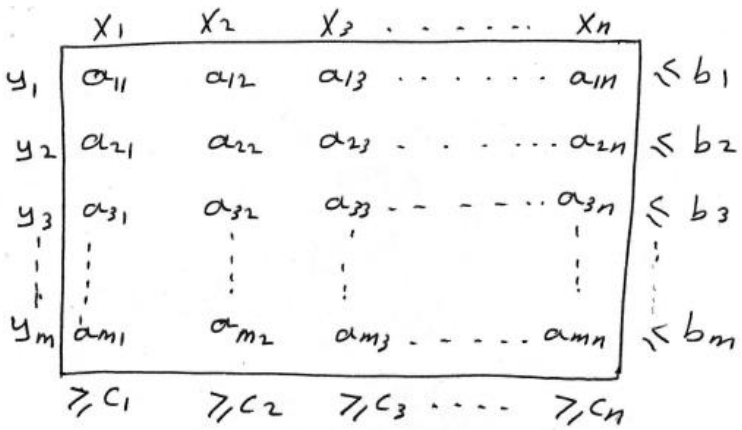
Subject to

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n \leq b_2$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n \leq b_m$$
$$x_1, x_2, \ldots, x_n \geq 0$$

$$\text{minimize } w = b_1y_1 + b_2y_2 + \cdots + b_my_m$$

Subject to

$$a_{11}y_1 + a_{21}y_2 + \cdots + a_{m1}y_m \geq c_1$$
$$a_{12}y_1 + a_{22}y_2 + \cdots + a_{m2}y_m \geq c_2$$
$$\vdots$$
$$a_{1n}y_1 + a_{2n}y_2 + \cdots + a_{mn}y_m \geq c_n$$
$$y_1, y_2, \ldots, y_m \geq 0$$



The above pair of programming can be written as:

Primal

$$\text{Max } z = \sum_{j=1}^{n} c_j x_j$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \le b_i, \quad i = 1,2,\dots,m$$

$$x_j \ge 0, \ j = 1,2,\dots,n$$

Dual

$$\text{Min } w = \sum_{i=1}^{m} b_i y_i$$

subject to

$$\sum_{i=1}^{m} a_{ij} y_i \ge c_j, \quad j = 1,2,\dots,n$$

$$y_i \ge 0, \quad i = 1,2,\dots,m$$

In matrix notation the symmetric dual LP is written as:

Primal

$$Max \ \ z = c^T x$$

s.t

$$Ax \le b$$

$$x \ge 0$$

Dual

$$Min \ \ w = b^T y$$

s.t

$$yA \ge c$$

$$y \ge 0$$

Where A is $m \times$ n matrix, b is $m \times 1$ column vector, x is $n \times 1$ column vector, y is $1 \times$ m row vector, and c is $1 \times$ n row vector.

Example 1: Consider the following primal LP.

$$\text{Max } z = x_1 + 2x_2 - 3x_3 + 4x_4$$

$$\text{s.t} \quad \begin{aligned} x_1 + 2x_2 + 2x_3 - 3x_4 &\le 25 \quad (y_1) \\ 2x_1 + x_2 - 3x_3 + 2x_4 &\le 15 \quad (y_2) \\ x_1, x_2, x_3, x_4 &\ge 0 \end{aligned}$$

Write the dual of the above problem

Solution:

$$\text{Min } w = 25y_1 + 15y_2$$

$$\text{s.t} \quad \begin{aligned} y_1 + 2y_2 &\ge 1 \\ 2y_1 + y_2 &\ge 2 \\ 2y_1 - 3y_2 &\ge -3 \\ -3y_1 + 2y_2 &\ge 4 \\ y_1, y_2 &\ge 0 \end{aligned}$$

y_1 and y_2 called the dual variables.

Example 2: Construct the dual of the following LP.

$$\text{Min } z = 3x_1 - 2x_2 + 4x_3$$

$$\begin{aligned} 3x_1 + 5x_2 + 4x_3 &\geq 7 \\ 6x_1 + x_2 + 3x_3 &\geq 4 \\ \text{s.t} \quad 7x_1 - 2x_2 - x_3 &\leq 10 \\ x_1 - 2x_2 + 5x_3 &\geq 3 \\ 4x_1 + 7x_2 - 2x_3 &\geq 2 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

Solution: multiply $7x_1 - 2x_2 - x_3 \leq 10 \quad (*-1)$

$$\text{Min } z = 3x_1 - 2x_2 + 4x_3$$

$$\begin{aligned} 3x_1 + 5x_2 + 4x_3 &\geq 7 && y_1 \\ 6x_1 + x_2 + 3x_3 &\geq 4 && y_2 \\ \text{s.t} \quad -7x_1 + 2x_2 + x_3 &\geq -10 && y_3 \\ x_1 - 2x_2 + 5x_3 &\geq 3 && y_4 \\ 4x_1 + 7x_2 - 2x_3 &\geq 2 && y_5 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

The dual of the given problem will be

$$\begin{aligned} \text{Max } w = 7y_1 + 4y_2 - 10y_3 + 3y_4 + 2y_5 \\ 3y_1 + 6y_2 - 7y_3 + y_4 + 4y_5 \leq 3 \\ 5y_1 + y_2 + 2y_3 - 2y_4 + 7y_5 \leq -2 \\ 4y_1 + 3y_2 + y_3 + 5y_4 - 2y_5 \leq 4 \\ y_1, y_2, y_3, y_4, y_5 \geq 0 \end{aligned}$$

Example 3: Construct the dual of the following LP.

$$\text{Max } z = 5x_1 + 6x_2$$

$$\begin{aligned} x_1 + 2x_2 &= 5 && y_1 \\ -x_1 + 5x_2 &\geq 3 && y_2 \\ \text{s.t} \quad 4x_1 + 7x_2 &\leq 8 && y_3 \end{aligned}$$

$$x_1 \text{ unrestricted in sign}, x_2 \geq 0$$

Solution: $-x_1 + 5x_2 \geq 3 \quad (*-1)$

So it becomes like this

$$\text{Max } z = 5x_1 + 6x_2$$

$$\begin{array}{lll} & x_1 + 2x_2 = 5 & y_1 \\ & x_1 - 5x_2 \leq -3 & y_2 \\ \text{s.t} & 4x_1 + 7x_2 \leq 8 & y_3 \end{array}$$

$$x_1 \text{ unrestricted in sign}, x_2 \geq 0$$

The dual is

$$\begin{array}{lll} \text{Min} & w = 5y_1 - 3y_2 + 8y_3 & \\ & y_1 + y_2 + 4y_3 = 5 & \\ \text{s.t} & 2y_1 - 5y_2 + 7y_3 \geq 6 & \\ & y_2, y_3 \geq 0, y_1 \text{ unrestricted in sign} & \end{array}$$

Maximization problem		Minimization problem
Constraints		***Variables***
$\geq$	$\Leftrightarrow$	≤ 0
$\leq$	$\Leftrightarrow$	≥ 0
$=$	$\Leftrightarrow$	Unrestricted
Variables		***Constraints***
≥ 0	$\Leftrightarrow$	$\geq$
≤ 0	$\Leftrightarrow$	$\leq$
Unrestricted	$\Leftrightarrow$	$=$

H.W: Find the dual for the following LP.

$$\text{Min } z = x_2 + 3x_3$$

$$\begin{array}{ll} & 2x_1 + x_2 \leq 3 \\ & x_1 + 2x_2 + 6x_3 \geq 5 \\ \text{s.t} & -x_1 + x_2 + 2x_3 = 2 \\ & x_1, x_2, x_3 \geq 0 \end{array}$$

$$\text{Min } z = 15x_1 + 12x_2$$

$$\begin{array}{ll} \text{s.t} & x_1 + 2x_2 \geq 3 \\ & 2x_1 - 4x_2 \leq 5 \\ & x_1, x_2 \geq 0 \end{array}$$